Design and Analysis of Algorithms

Unit - II

Dr. R. Bhuvaneswari
Assistant Professor
Department of Computer Science
Periyar Govt. Arts College, Cuddalore.

Periyar Govt. Arts College
Cuddalore

Divide and Conquer

Syllabus

UNIT - II: DIVIDE AND CONQUER

General Method - Binary Search - Finding the Maximum and Minimum - Merge Sort - Quick Sort - Selection Sort - Strassen's Matrix Multiplications.

TEXT BOOK

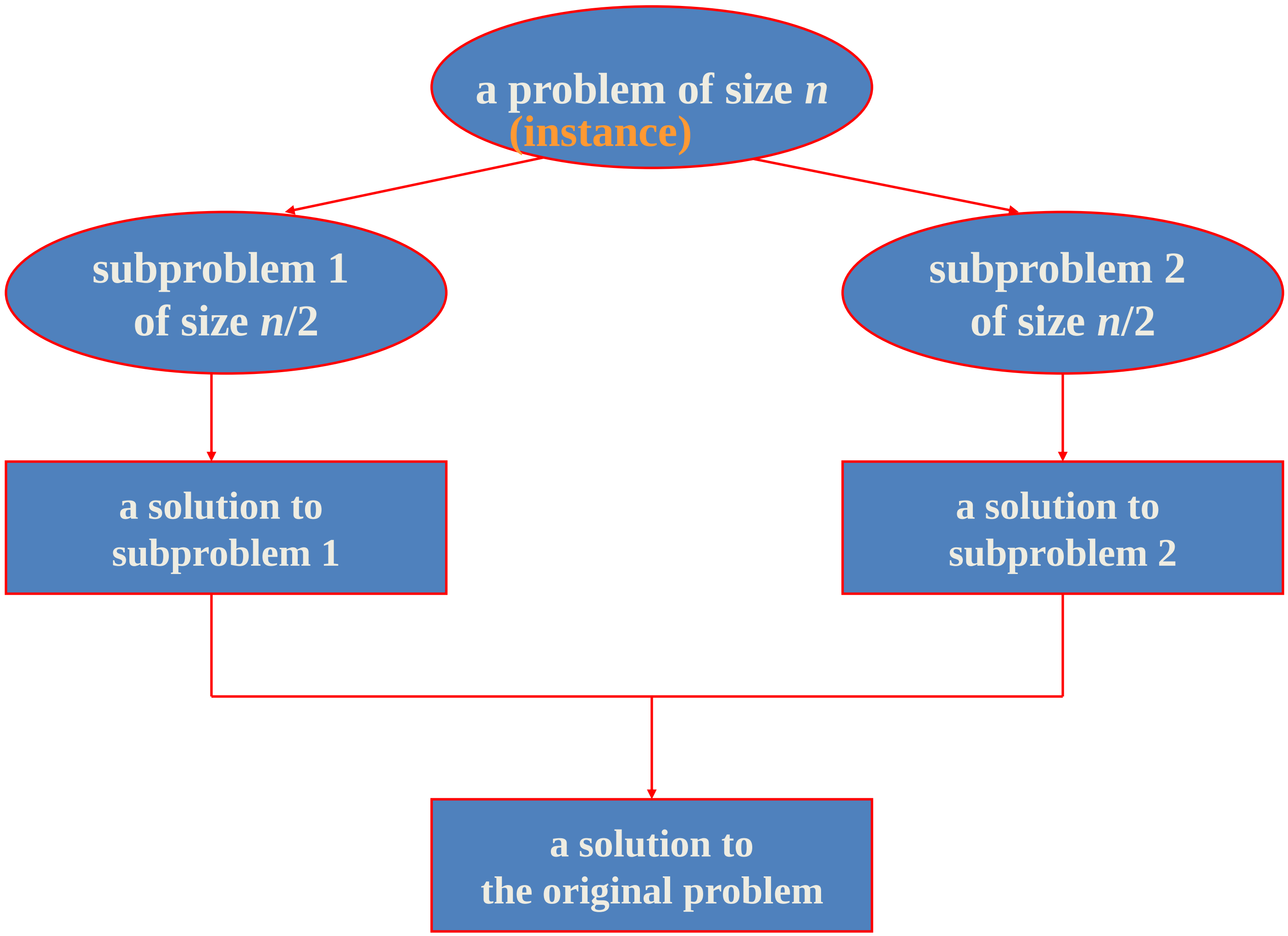
Fundamentals of Computer Algorithms, Ellis Horowitz, Sartaj Sahni, Sanguthevar Rajasekaran, Galgotia Publications, 2015.

Divide and Conquer

General Method:

- Given a function to compute on 'n' inputs, divide-and-conquer strategy suggests splitting the inputs into 'k' distinct subsets, $1<k\leq n$, yielding 'k' **subproblems**.
- These subproblems must be solved, and then a method must be found to combine sub solutions into a solution of the whole.
- If the subproblems are still relatively large, then the divide-and-conquer strategy can possibly be reapplied.
- For those cases the re-application of the divide-and-conquer principle is naturally expressed by a recursive algorithm.

Divide and Conquer



Divide and Conquer

Control Abstraction of Divide and Conquer

```
Algorithm DAndC(P)
{
if small(P) then
     return S(P);
else
{
    divide P into smaller instance P1, P2…….., Pk, k≥1;
    apply DAndC to each of these subproblems;
    return combine(DAndC(P1), DAndC(P2), ……., DAndC(Pk));
}
}
```

Divide and Conquer

Computing time of DAndC is:

$$T(n) = \begin{cases} g(n) & n \text{ small} \\ T(n1) + T(n2) + \ldots + T(nk) + f(n) & \text{otherwise} \end{cases}$$

where

T(n) is the time for DAndC on any input of size n

g(n) is the time to compute the answer directly for small inputs

f(n) is the time for dividing P and combining the solutions to subproblems

Binary Search

The following concept is used to search an element in the given array:

- Find the middle element
- Check the middle element with the element to be found.
- If the middle element is equal to that element, then it will provide the output.
- If the value is not same, then it will check whether the middle element value is less than or greater than the element to be found.
- If the value is less than that element, then the search will start with the elements next to the middle element.
- If the value is high than that element, then the search will start with the elements before the middle element.
- This process continues, until that particular element has been found.

Binary Search

- Let a_i be a list of elements that are in non-decreasing order. $1 \leq i \leq n$.
- It is a problem of determining whether a given element x is present in the list.

1	2	3	4	5	6	7	8	9	10
10	20	30	40	50	60	70	80	90	100

mid = $\lfloor (low+high)/2 \rfloor$

x = 60

1. low = 1, high = 10
   mid = (1+10)/2 = 5, 60 > 50, low = 6
2. low = 6, high = 10
   mid = (6 + 10)/2 = 8, 60 < 80, high = 7
3. low = 6, high = 7
   mid = (6 + 7)/2 = 6

1. (x<a[mid]) then
   high = mid-1
2. else if (x>a[mid]) then
   low = mid+1
3. else return mid;

Binary Search

1	2	3	4	5	6	7	8	9	10
10	20	30	40	50	60	70	80	90	100

```
Algorithm BinSearch(a,n,x)
//Given an array a[1:n] of elements in
//nondecreasing order, n ≥ 0
{
   low = 1; high = n;
   while (low ≤ high) do
   {
      mid := ⌊(low+high)/2⌋;
      if (x < a[mid]) then high = mid-1;
      else if (x > a[mid]) then low := mid+1;
            else return mid;
   }
   return 0;
}
```

mid = (low+high)/2 **x = 30**

1. low = 1, high = 10
   mid = (1+10)/2 = 5, 30 < 50, high = 4
2. low = 1, high = 4
   mid = (1 + 4)/2 = 2, 30 > 20, low = 3
3. low = 3, high = 4
   mid = (3 + 4)/2 = 3

Binary Search using recursion

1	2	3	4	5	6	7	8	9	10
10	20	30	40	50	60	70	80	90	100

```
Algorithm BinSrch(a,i,l,x)
{
if (l = i) then
{
   if (x = a[i]) then return i;
   else return 0;
}
else
{
   mid = ⌊(i+l)/2⌋;
   if (x = a[mid]) then return mid;
   else if (x < a[mid]) then return BinSrch(a,i,mid-1,x);
         else return BinSrch(a,mid+1,l,x);
}
}
```

mid = (i+l)/2 **x = 30**

1. i = 1, l = 10
   mid = (1+10)/2 = 5, 30 < 50, l = 4
2. i = 1, l = 4
   mid = (1 + 4)/2 = 2, 30 > 20, i = 3
3. i = 3, l = 4
   mid = (3 + 4)/2 = 3

Binary Search

Time Complexity

1. If the search element is the middle element of the array, **in this case, time complexity will be O(1), the best case.**
2. Otherwise, binary search algorithm breaks the array into half in each iteration.

The array is divided by 2 until the array has only one element:

$$\frac{n}{2^k} = 1$$

we can rewrite it as:

$$n = 2^k$$

by taking log both side, we get

$$\log_2^n = \log_2 2^k$$

$$\log_2^n = k\log_2^2$$

$$k = \log_2^n \text{ (since } \log_a^a = 1)$$

The time complexity of binary search is $\log_2^n$

Finding the maximum and minimum

- The problem to find the maximum and minimum items in a set of n elements.

Algorithm StraightMaxMin(a,n,max,min)

```
// set max to maximim and min to the
// minimum of a[1:n]
{
   max := min := a[1];
   for i := 2 to n do
   {
     if (a[i] > max) then max := a[i];
     if (a[i] < min) then min := a[i];
   }
}
```

- StraightMaxMin requires 2(n-1) element comparisons in the best, average and worst cases.

1	2	3	4	5
37	78	45	12	92

max = min = 37
i = 2
max = 78; min = 37
i = 3
max = 78; min = 37
i = 4
max = 78; min =12
i = 5
max = 92; min = 12

Finding the maximum and minimum

- An immediate improvement is possible by realizing that the comparison a[i] < min is necessary only when a[i] > max is false.

  Hence we can replace the contents of the for loop by

  ```
  if (a[i] > max) then max := a[i];
  else if (a[i] < min) then min := a[i];
  ```
- When the elements are in the increasing order the number of element comparisons is n-1.
- When the elements are in the decreasing order the number of element comparisons is 2(n-1).

Finding the maximum and minimum

Divide and Conquer Algorithm

- Let P = (n, a [i],……,a [j]) denote an arbitrary instance of the problem.
- Here ‘n’ is the no. of elements in the list (a[i],….,a[j]) and we are interested in finding the maximum and minimum of the list.
- If the list has more than 2 elements, P has to be divided into smaller instances.
- We divide ‘P’ into the 2 instances,
  - P1=([n/2], a[1],……..a[n/2]) and
  - P2= (n-[n/2], a[[n/2]+1],….., a[n])
- After having divided ‘P’ into 2 smaller sub problems, we can solve them by recursively invoking the same divide-and-conquer algorithm.
- max(P) is the maximum of max(P1) and max(P2)
- min(P) is the minimum of min(P1) and min(P2)

Finding the maximum and minimum

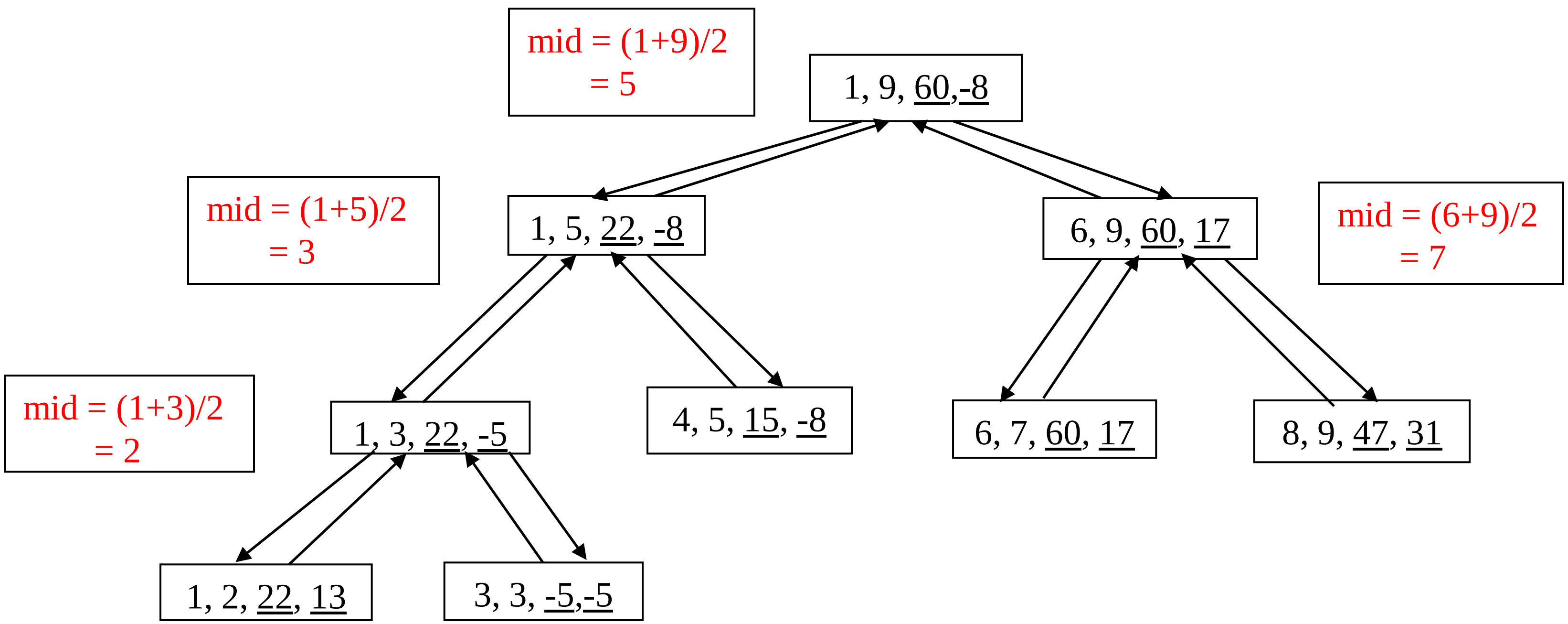
```
Algorithm MaxMin(i,j,max,min)
//a[1:n] is a global array.
{
if (i = j) then max = min = a[i];
else if (i = j-1) then
{
  if (a[i] < a[j]) then
  {
     max = a[j]; min = a[i];
  }
  else
  {
     max = a[i]; min = a[j];
  }
}
else
{
  mid = ⌊(i+j)/2⌋;
   MaxMin(i,mid,max,min);
   MaxMin(mid+1,j,max1,min1);
   if(max < max1) then max = max1;
   if (min > min1) then min = min1;
  }
}
```

Finding the maximum and minimum

Example: find max and min in the array:

22, 13, -5, -8, 15, 60, 17, 31, 47 (n = 9)

Index:	1	2	3	4	5	6	7	8	9
Array:	22	13	-5	-8	15	60	17	31	47



Finding the maximum and minimum

The number of element comparisons T(n) is represented as recurrence relation

$$T(n) = \begin{cases} T\left(\frac{n}{2}\right) + T\left(\frac{n}{n}\right) + 2 & n > 2 \\ 1 & n = 2 \\ 0 & n = 1 \end{cases}$$

When n is a power of two, $n = 2^k$ for some positive integer k, then

$$\begin{aligned} T(n) &= 2T(n/2)+2 \\ &= 2(2T(n/4)+2)+2 \\ &= 4T(n/4) + 4 + 2 \\ &= 4(2T(n/8) + 2) + 4 + 2 \\ &= 8T(n/8) + 8 + 4 + 2 \\ &\quad \ldots\ldots. \\ &= 2^k T(n/2^k) + 2^k + 2^{k-1} + 2^{k-2} + \ldots\ldots + 2 \end{aligned}$$

Finding the maximum and minimum

Taking T(2) = 1

ie. $\frac{n}{2^k} = 2$

$T(n) = 2^k + 2^k + 2^{k-1} + 2^{k-2} + \ldots\ldots. + 2$

$= 2^k + \sum_{j=1}^{k} 2^j$

$= 2^k + 2 * \frac{(2^k - 1)}{2-1}$

$= \frac{n}{2} + 2 * (\frac{n}{2} - 1)$

$= \frac{n}{2} + n - 2$

$= \frac{3n}{2} - 2$

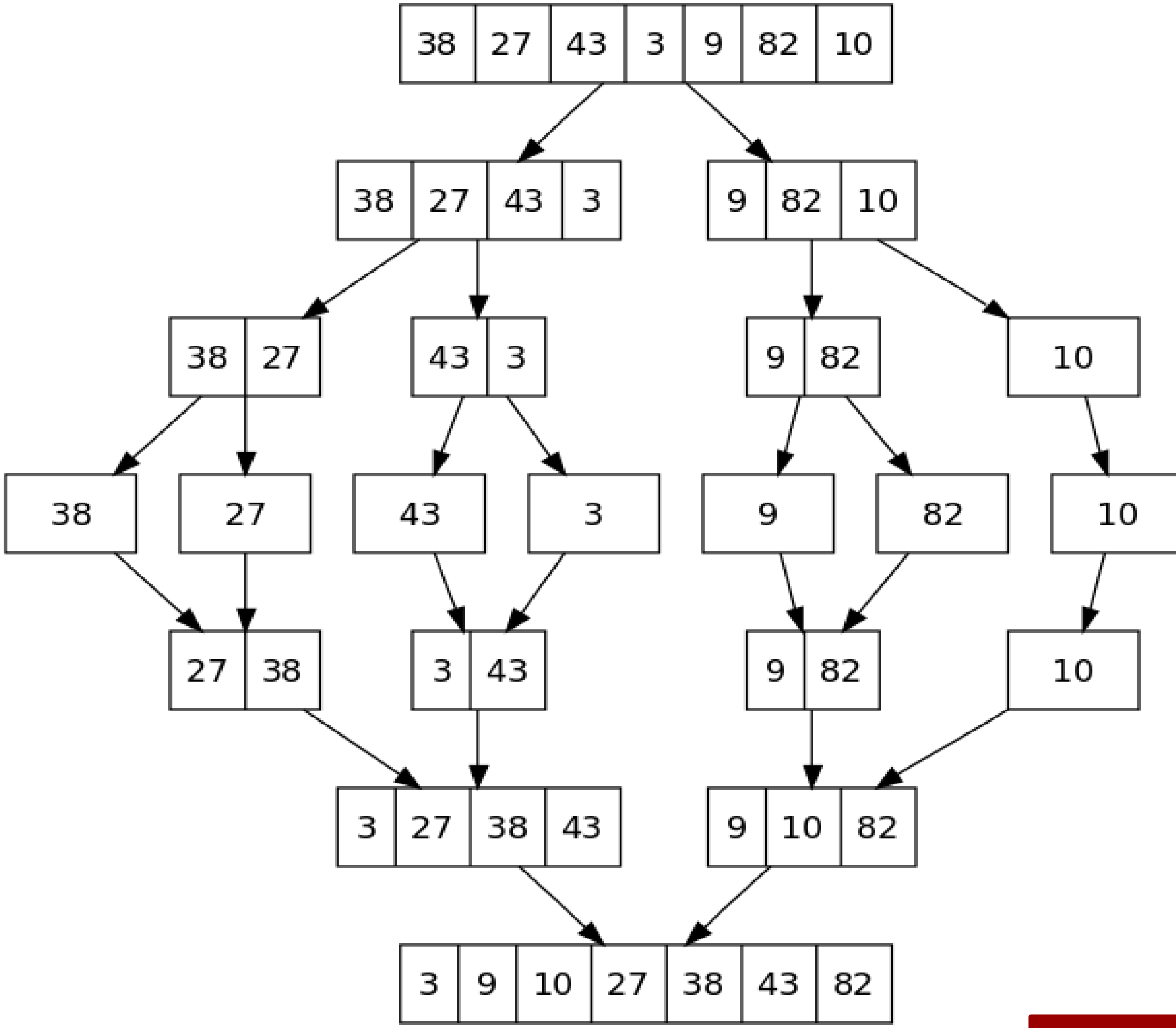
$$Since, \sum_{j=1}^{n} x^j = x * \frac{x^n - 1}{x - 1}$$

Therefore, 3n/2- 2 is the best, average and worst case number of comparisons where n is power of 2.

Merge Sort

- **Sort** a sequence of n elements into non-decreasing order.
- Merge sort is a sorting technique based on divide and conquer technique.
- Merge sort first divides the unsorted list into two equal halves.
- Sort each of the two sub lists recursively until we have list size of length 1, in which case the list itself is returned.
- Merge the two sorted sub lists back into one sorted list.

Merge Sort

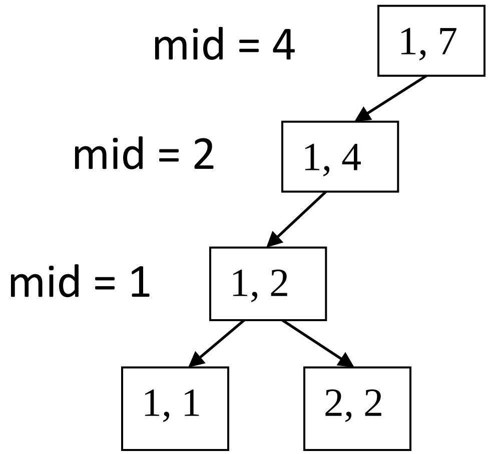


38 27 43 3 9 82 10
38 27 43 3
9 82 10
38 27
43 3
9 82
10
38
27
43
3
9
82
10
27 38
3 43
9 82
10
3 27 38 43
9 10 82
3 9 10 27 38 43 82

Merge Sort

```
Algorithm MergeSort(low,high)
{
 If (low < high) then
 {
   mid = ⌊(low+high)/2⌋;
   MergeSort(low,mid);
   MergeSort(mid+1,high);
   Merge(low,mid,high);
 }
}
```

38	27	43	3	9	82	10



Merge Sort

```
Algorithm Merge(low,mid,high)
//b[] is an auxiliary global array.
{
  h=low; i=low; j=mid+1;
  while((h≤mid) and (j≤high)) do
  {
     if(a[h] ≤ a[j]) then
     {
        b[i] = a[h]; h = h+1;
     }
     else
     {
        b[i] = a[j]; j = j+1;
     }
     i = i+1;
  }
  if(h > mid) then
  {
     for k = j to high do
     {
        b[i] = a[k]; i = i+1;
     }
  }
  else
  {
     for k = h to mid do
     {
        b[i] = a[k]; i = i+1;
     }
  }
  for k = low to high do
        a[k] = b[k];
}
```

Merge Sort

Computing time for merge sort is described by the recurrence relation,

$$T(n) = \begin{cases} a & n = 1, a \text{ is a constant} \\ 2T\left(\frac{n}{2}\right) + cn & n > 1, c \text{ is a constant} \end{cases}$$

when n = 2^k

$$\begin{aligned} T(n) &= 2T(n/2) + cn \\ &= 2[2T(n/4) + cn/2] + cn \\ &= 4T(n/4) + cn + cn \\ &= 4T(n/4) + 2cn \\ &= 4[2T(n/8) + cn/4] + 2cn \\ &= 8T(n/8) + cn + 2cn \\ &= 8T(n/8) + 3cn \\ &\quad \ldots\ldots\ldots\ldots\ldots \\ &= 2^kT(n/2^k) + kcn \\ &= 2^kT(1) + \mathbf{k}cn \\ &= an + cn\log n \end{aligned}$$

Since,

$T(n/2^k = 1)$

$n = 2^k$

$$\begin{aligned} \log_2{}^n &= \log_2 2^k \\ &= k\log_2{}^2 \\ &= k \end{aligned}$$

Quick Sort

- In merge sort, the array a[1:n] was divided at its midpoint into sub arrays which were independently sorted and later merged.
- In quick sort, the division into 2 sub arrays is made so that the sorted sub arrays do not need to be merged later.
- This is accomplished by rearranging the elements in a[1:n] such that $a[i] \leq a[j]$ for all i between 1 and m and all j between (m+1) and n for some m, $1 \leq m \leq n$.
- Thus the elements in a[1:m] and a[m+1:n] can be independently sorted.
- No merging is needed.
- This rearranging is referred to as partitioning.

Quick Sort

- Quick sort picks an element as pivot element and partitions the given array around the picked pivot.
- There are many different versions of quick sort that pick pivot in different ways.
  - ➢ pick first element as pivot.
  - ➢ pick last element as pivot.
  - ➢ Pick a random element as pivot.
  - ➢ Pick median as pivot.
- The role of the pivot value is to assist with splitting the list.
- The actual position where the pivot value belongs in the final sorted list, commonly called the **split point**, will be used to divide the list for subsequent calls to the quick sort.

Quick Sort Example

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	i	j
65	70	75	80	85	60	55	50	45	$+\infty$	2	9
65	45	75	80	85	60	55	50	70	$+\infty$	3	8
65	45	50	80	85	60	55	75	70	$+\infty$	4	7
65	45	50	55	85	60	80	75	70	$+\infty$	5	6
65	45	50	55	60	85	80	75	70	$+\infty$	6	5
60	45	50	55	65	85	80	75	70	$+\infty$		

Quick Sort

```
Algorithm Quicksort(p,q)
{
if (p<q) then
  {
     j:= Partititon (a,p,q+1);
     Quicksort(p,j-1);
     Quicksort(j+1,q);
  }
}

Algorithm Partition(a,m,p)
{
  v:=a[m]; i:=m; j:=p;
  repeat
 {
    repeat
         i:=i+1;
    until (a[i] ≥ v);
    repeat
         j := j-1;
    until (a[j] ≤ v);
    if (i < j) then Interchange(a, i, j);
 }until ( i ≥j);
 a[m] := a[j];
 a[j] := v;
 return j;
}

Algorithm Interchange(a, i, j)
{
  p := a[i];
  a[i] := a[j];
  a[j] := p;
}
```

Quick Sort

Computing time for Quick sort

$$T(n) = 2T(n/2) + n \text{ for } n > 1, \qquad T(1) = 0$$

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2[2T(n/4) + n/2] + n \\
&= 4T(n/4) + n + n \\
&= 4T(n/4) + 2n \\
&= 4[2T(n/8) + n/4] + 2n \\
&= 8T(n/8) + n + 2n \\
&= 8T(n/8) + 3n \\
&\quad \ldots\ldots\ldots\ldots\ldots \\
&= 2^k T(n/2^k) + kn \\
&= nT(1) + kn \\
&= n\log n
\end{aligned}
$$

Since,

$T(n/2^k = 1)$

$n = 2^k$

$$
\begin{aligned}
\log_2{}^n &= \log_2 2^k \\
&= k\log_2{}^2 \\
&= k
\end{aligned}
$$

Selection Sort

- Selection sort is the most simplest sorting algorithm.
- Following are the steps involved in selection sort(for sorting a given array in ascending order):
  - Starting from the first element, search the smallest element in the array, and replace it with the element in the first position.
  - Then move on to the second position, and look for smallest element present in the subarray, starting from index 2 till the last index.
  - Replace the element at the **second** position in the original array with the second smallest element.
  - This is repeated, until the array is completely sorted.

Selection Sort Example

42	16	84	12	77	26	53	The array, before the selection sort operation begins.
12	16	84	42	77	26	53	The smallest number (**12**) is swapped into the first element in the structure.
12	16	84	42	77	26	53	In the data that remains, **16** is the smallest; and it does not need to be moved.
12	16	26	42	77	84	53	**26** is the next smallest number, and it is swapped into the third position.
12	16	26	42	77	84	53	**42** is the next smallest number; it is already in the correct position.
12	16	26	42	53	84	77	**53** is the smallest number in the data that remains; and it is swapped to the appropriate position.
12	16	26	42	53	77	84	Of the two remaining data items, **77** is the smaller; the items are swapped. *The selection sort is now complete.*

Selection Sort

```
Algorithm Selection(a, n)
{
  for i := 1 to n-1 do
 {
   min := a[i];
   loc := i;
   for j := i+1 to n do
   {
     if (min > a[j] ) then
     {
        min := a[j];
        loc :=j;
     }
   }
   temp := a[i]; a[i] := a[loc]; a[loc] := temp;
 }
}
```

Selection Sort

Number of comparisons in selection sort:

$(n-1) + (n-2) + (n-3) + \ldots\ldots + 2 + 1$

$n(n-1)/2$ comparisons

Strassen's Matrix Multiplication

- Let A and B be two n x n matrices.
- The product matrix C = AB is also an n x n matrix whose i, j^{th} element is formed by taking the elements in the i^{th} row of A and j^{th} column of B and multiplying them to get

$$C(i,j) = \sum_{1 \le k \le n} A(i,k)B(k,j) \quad \text{for all i and j between 1 and n.}$$

- To compute C(i,j) using this formula, we need n multiplications.
- As the matrix C has n^2 elements, the time for the resulting matrix multiplication algorithm is **$O(n^3)$**.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$C_{11} = A_{11} * B_{11} + A_{12} * B_{21}$

$C_{12} = A_{11} * B_{12} + A_{12} * B_{22}$

8 multiplications and 4 additions

$C_{21} = A_{21} * B_{11} + A_{22} * B_{21}$

$C_{22} = A_{21} * B_{12} + A_{22} * B_{22}$

Strassen's Matrix Multiplication

- Divide and conquer method suggests another way to compute the product of two n x n matrices.
- We assume that n is a power of 2.
- If n is not a power of two, then enough rows and columns of zeros can be added to both A and B so that the resulting dimensions are power of 2.
- If n = 2, conventional matrix multiplication is performed.
- If n > 2, then the elements are partitioned into sub matrix n/2 x n/2.
- Since n is power of 2, these matrix products can be recursively computed by the same algorithm we are using for the n x n case.
- The overall computing time T(n) of the resulting divide-and-conquer algorithm is given by the recurrence

$$T(n) = \begin{cases} b & n \leq 2 \\ 8T\left(\frac{n}{2}\right) + cn^2 & n > 2 \end{cases}$$

where b and c are constants.

Strassen's Matrix Multiplication

- Strassen showed that 2 x 2 matrix multiplication can be done in 7 multiplications and 18 additions or subtractions.
- This reduce can be done by divide and conquer approach.

$$P = (A_{11} + A_{22})(B_{11} + B_{22})$$
$$Q = (A_{21} + A_{22})B_{11}$$
$$R = A_{11}(B_{12} - B_{22})$$
$$S = A_{22}(B_{21} - B_{11})$$
$$T = (A_{11} + A_{12})B_{22}$$
$$U = (A_{21} - A_{11})(B_{11} + B_{12})$$
$$V = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$C_{11} = P + S - T + V$$
$$C_{12} = R + T$$
$$C_{21} = Q + S$$
$$C_{22} = P + R - Q + U$$

The resulting recurrence relation for T(n) is

$$T(n) = \begin{cases} b & n \leq 2 \\ 7T\left(\frac{n}{2}\right) + an^2 & n > 2 \end{cases}$$

where a and b are constants. $T(n) = \mathbf{O(n^{2.81})}$